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2005 J. Phys. A: Math. Gen. 38 10843

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Ballistic dynamics of a convex smooth-wall billiard with finite escape rate along the boundary

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Received 8 July 2005, in final form 27 September 2005

Published 22 November 2005

Online at stacks.iop.org/JPhysA/38/10843

Abstract

We focus on the problem of an impurity-free billiard with a random position-dependent boundary coupling to the environment. The response functions of such an open system can be obtained non-perturbatively from a supersymmetric generating functional. The derivation of this functional is based on averaging over the escape rates and results in a nonlinear ballistic σ -model, characterized by system-specific parameters. Particular emphasis is placed on the ‘whispering gallery modes’ as the origin of surface diffusion modes in the limit of large dimensionless conductance.

PACS number: 05.45.Mt

1. Introduction

Wave billiards with smooth boundary walls find a wide variety of applications in condensed matter physics [1], microwave and acoustic chaology [2]. For chaotic transport in nano-size two-dimensional electronic systems such as quantum dots, wires, junctions and corrals, billiards serve as convenient theoretical models for a confined electronic gas. Chaos in a cavity might be attributed to the presence of impurities or to the properties of the boundary itself. A review of closed billiard dynamics can be found in [3]. Convex hard-wall cavities open to the environment are also well studied within the semiclassical approximation [4].

An intermediate situation arises when one allows for a small but finite coupling to the outside world all around the boundary. Furthermore, if this coupling is described by a random set of coupling coefficients to a large number of ideal leads, the resulting system acquires a natural statistical description [5]. Indeed, in studies of nanostructures one typically aims at the statistical properties of various response functions, computed through the cavity Green’s function. In experiments, the statistics are obtained via spectral averaging or from an ensemble of different boundary configurations, corresponding to the same spectral density. In both regular and chaotic cavities, the most well-studied statistical ensemble is the one defined over the different configurations of the internal impurities [6–9].

The notion of a statistical ensemble does not come naturally for the analytical study of a billiard with no internal scattering potential, no magnetic impurities or random magnetic field. The scattering, which in this case takes place exclusively at the boundary, is the source of stochasticity. In the simplest case the scattering is specular, but it can also be diffusive, as in [10]. Furthermore, the scattering may be accompanied by the escape of scattered waves into the exterior [5], as, for example, in quantum dots [11] or quantum corrals [12]. In this paper, we present a more detailed description of the method proposed in [5]. The model system we study is a billiard weakly attached to the large number of ideal infinite leads [5]. For simplicity and clarity of presentation we consider one open channel in each waveguide, with a coupling coefficient which is Gaussian distributed around a small, but non-zero mean. The mean and the width of the coupling coefficient distribution are assumed to vary smoothly with the position of the lead on the perimeter.

For this dot, we construct the supersymmetric generating functional [6, 9] and performing the average over the ensemble of realizations of coupling coefficients, we obtain a ‘surface’ nonlinear σ -model (NL σ M). Its ‘diffusion’ modes are confined to the boundary of the dot. In convex nearly closed billiards these correspond to the so-called whispering gallery modes (WGM), which are high angular momentum modes corresponding to the classical trajectories running alongside the billiard walls. The WGM are exponentially less likely to escape compared to the modes with incidence directions close to the lead normals, as can be inferred, for example, from [13]. Thus, the response functions at very large times are expected to be dominated by these modes, and hence can be non-perturbatively calculated from our supersymmetric functional.

Some of the technical tools we employ, such as the non-Hermitian effective Hamiltonian, have been frequently used along with random matrix theory (RMT) [14–16]. The use of RMT implies a restriction to the universal regime. Here, we pursue a description of the nonuniversal regime of the billiard dynamics, thus covering a much broader energy range than is possible with RMT. We are also interested in spatially dependent characteristics of the system, inaccessible via RMT.

The assumption of only one open channel is not essential. The generalization of our approach to an arbitrary number of open channels is straightforward. The choice of random Gaussian distribution of the channel strengths is in fact quite realistic; the actual values of couplings (and the number of open channels) are not known in general [17]. It is usually argued, at least in situations where the RMT is applicable, that quantitative results do not depend on the coupling being random or constant [14]. For our development, however, the randomness is essential as we briefly explained above.

We note that our model applies, *mutatis mutandis*, to a system with local or extended sources of damping [17, 18]. Besides quantum dots [11], there are other mesoscopic billiards which fall into this category, such as quantum and optical corrals [12, 19], optical resonant cavities [20] and the artificial atoms proposed in [21]. Here also we expect the WGM to play an important role in the long-time Green’s functions. The dependence of escape rate on the angular momentum, important in both optical and electronic systems, is also incorporated in our model.

Finally, weak interactions in nanostructures with large dimensionless conductance creates an additional motivation for our study. For a ballistic cavity, in which electronic interactions are of interest, while the shape is not, the convex smooth-wall billiard represents a sufficient starting point. It might be possible to use the results of our present analysis in the large- N approach of [22] to the interacting ballistic case. We also note in passing that the fundamental problem of constructing a σ -model for a closed, impurity-free ballistic billiard has proven to be technically challenging [10, 23–25]. Our work shows that a random coupling to the

environment acts as a natural regularizer and allows us to circumvent many of the technical difficulties.

The plan of this paper is as follows. Section 2 reviews the procedure for ‘integrating out’ the leads to arrive at a non-Hermitian Hamiltonian which describes electronic scattering in a quantum dot. In section 3, we construct the supersymmetric generating functional for the correlation functions and carry out the ensemble average, introducing a supermatrix field for each lead to decouple supervector fields. The continuous model resulting in the limit of large leads number is analysed in section 4. We demonstrate how a Born–Oppenheimer-like approximation [26] can be employed in the solution of the effective problem, which results from the saddle point evaluation of superintegral. This is done in section 5. We complete the derivation of the surface diffusion NL σ M in section 6 and present our summary in section 7.

2. Effective non-Hermitian Hamiltonian

It is common in mesoscopic physics to study open systems using their closed counterparts and vice versa. The eigenfunction structure of closed dots affects the transport properties of the open dots, while the leads attached to the dot can change the nature of the dynamics from regular to chaotic [27]. In the RMT context, the scattering approach [14] to mesoscopic billiards relies on the decomposition of the system into internal and external subparts. The internal is described by N bound states, while the external by M channel states propagating along the hard wall leads to infinity. To set up the analysis one introduces (i) the $N \times N$ internal Hamiltonian H_{in} , (ii) $M \times M$ S -matrix, which relates incoming and outgoing wave amplitudes in the asymptotic region:

$$\psi_{i,n} = \delta_{ji} \psi_{i,n}^{\text{incoming}} + \sum_{m=1}^{M_j} S_{mn}^{(ji)} \psi_{j,m}^{\text{outgoing}},$$

where i, j specify leads and m, n specify channels and (iii) express S -matrix in terms of H_{in} and $N \times M$ matrix W , which couples the subparts $S = I - 2\pi i W^\dagger (E - H_{\text{eff}})^{-1} W$, with $H_{\text{eff}} = H_{\text{in}} - i\pi W W^\dagger$. This relation of S -matrix to the effective Green’s function $(E - H_{\text{eff}})^{-1}$ is the building block of the Hamiltonian approach to the system’s statistics in the universal (RMT) regime, i.e., the regime independent of the details of the underlying classical dynamics [14]. The non-Hermitian random matrices have been the subject of numerous works on quantum chaotic scattering (see, for example, [16] and references therein).

Similar steps lead to the generalization of the effective non-Hermitian Hamiltonian in the nonuniversal regime, as was done in a thick wire in [28] and for a disordered quantum dot, in which electrons confined by a hard wall potential can escape into leads, in [29].

In full analogy with the RMT problem one can take the same approach to our problem involving open ballistic dot. For completeness, we reproduce the main steps carried out in [29], noting that we deal with a two-dimensional clean electronic system.

In order to study the Green’s functions G

$$(E \pm i\epsilon - \mathcal{H}_0) G^{R,A}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (1)$$

corresponding to the dot-plus-leads Hamiltonian

$$\mathcal{H}_0 = \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2, \quad (2)$$

we introduce the cross-sectional surfaces C_n perpendicular to the walls of the leads close to the place of their attachment to the dot. Then, we reformulate the problem by specifying the boundary conditions on these surfaces, thus eliminating the leads at the expense of modifying

the Hamiltonian of the system. First, we introduce auxiliary retarded and advanced Green's functions $\overline{G}^{R,A}(\mathbf{r}, \mathbf{r}')$ for each lead, satisfying equation (1) inside the region of the lead bounded by the section C_n . The boundary condition at C_n is the only difference between $\overline{G}^{R,A}(\mathbf{r}, \mathbf{r}')$ and $G^{R,A}(\mathbf{r}, \mathbf{r}')$. The former vanishes at this boundary. Considering $G^{R,A}(\mathbf{r}, \mathbf{r}')$, such that \mathbf{r} and \mathbf{r}' belong to the regions on the different sides from the section C_n , we make use of the following identity (current conservation):

$$\nabla_r [(\overline{G}^A(\mathbf{r}, \mathbf{r}'')^* \mathbf{v}_r G^R(\mathbf{r}, \mathbf{r}') + G^R(\mathbf{r}, \mathbf{r}') (\mathbf{v}_r \overline{G}^A(\mathbf{r}, \mathbf{r}'')^*)] = 2i\delta(\mathbf{r} - \mathbf{r}'')G^R(\mathbf{r}, \mathbf{r}'), \quad (3)$$

where $\mathbf{v}_r = -i(\nabla_r + e\mathbf{A}/c)/m$. To arrive at the current conservation relation (equation (3)), we combine definitions of both G^R and \overline{G}^A (equation (1)), by multiplying them with $G^R(\mathbf{r}, \mathbf{r}')$ and subtracting from each other. Then, integration with respect to \mathbf{r} -coordinates taken over the volume of the lead results in

$$G^R(\mathbf{r}'', \mathbf{r}') = -\frac{i}{2} \int_{C_n} (\mathbf{x}_n \cdot \mathbf{v}_r) (\overline{G}^A(\mathbf{r}, \mathbf{r}'')^* G^R(\mathbf{r}, \mathbf{r}')) \, d\mathbf{r}, \quad (4)$$

upon application of boundary condition on $\overline{G}^A(\mathbf{r}, \mathbf{r}'')$. Here, \mathbf{x}_n is the unit normal to C_n . Next, we apply the velocity operator $\mathbf{v}_{r''}$ to both sides of equation (4) and pick $\mathbf{r}'' = \mathbf{y}$ on the cross section C_n to get

$$(\mathbf{x}_n \cdot \mathbf{v}_y) G^R(\mathbf{y}, \mathbf{r}') = \int_{C_n} B_n(\mathbf{y}, \mathbf{r}) G^R(\mathbf{r}, \mathbf{r}') \, d\mathbf{r}, \quad (5)$$

where

$$B_n(\mathbf{y}, \mathbf{r}) = \frac{i}{2} (\mathbf{x}_n \cdot \mathbf{v}_y) (\mathbf{x}_n \cdot \mathbf{v}_r) \overline{G}^R(\mathbf{y}, \mathbf{r}),$$

and we used $(\overline{G}^A(\mathbf{r}, \mathbf{r}'')^*)^* = \overline{G}^R(\mathbf{r}'', \mathbf{r})$. Note, that the functions $B_n(\mathbf{y}, \mathbf{r})$ are completely determined by the properties of the leads, which we assumed to be ideal. Thus, the 'dot-plus-leads' system governed by equation (1) is now reduced to the dot only, governed by the same equation (1) and boundary conditions specified by equation (5). We also note that it is real part of a function $B_n(\mathbf{y}, \mathbf{r})$ that is related to the flux carried into the leads by the electronic states. To simplify the boundary conditions, we modify the Hamiltonian of the system even further according to

$$H = \mathcal{H}_0 \mp \frac{i}{2} \sum_{n=1}^N \widehat{B}_n \delta C_n, \quad (6)$$

where δC_n is a surface δ -function, defined via $\int \delta C_n \Psi(\mathbf{r}) \, d\mathbf{r} = \int_{C_n} \Psi(\mathbf{r}) \, d\mathbf{r}$, and $\widehat{B}_n \Psi(r) = \int_{C_n} \mathfrak{B} B_n(\mathbf{r}, \mathbf{r}') \Psi(\mathbf{r}') \, d\mathbf{r}'$, with $\mathfrak{B} B_n(\mathbf{r}, \mathbf{r}') = \gamma_n v_n \varphi_n(\mathbf{r}) \varphi_n(\mathbf{r}')$. To compactify the formulae of the next sections we also assume that each of the N leads contains at most one open channel. Thus, we end up looking for the Green's functions of the effective problem

$$(E - H)G^{R,A}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (7)$$

associated with the Neumann boundary conditions

$$(\mathbf{x}_n \cdot \mathbf{v}_r) G^{R,A}(\mathbf{r}, \mathbf{r}')|_{C_n+0} = 0, \quad (8)$$

where the derivatives are taken from the side of the lead (which is indicated by $C_n + 0$). The second term in Hamiltonian (equation (6)) describes the finite escape probability for the electrons colliding with the boundary. Now we are ready to proceed with the construction of the σ -model along the conventional lines developed for closed systems [9]. Note that boundary condition given by equation (8) ensures that system is effectively closed, i.e. component of the current, which is normal to the boundary, vanishes.

3. Supersymmetric generating functional

Next we construct the generating functional for both retarded and advanced Green's functions for Gaussian distributed dimensionless coupling coefficients γ_n . These coefficients are related to sticking probabilities and transmission coefficients, frequently used within the Hamiltonian approach to chaotic scattering [14] to compute statistical distributions of the resonance widths, delay times and related characteristics. A detailed discussion of the physical meaning of these coefficients can be found in [2, 17, 18, 30] and in references therein. In order to perform non-perturbative calculations of these averages in the nonuniversal regime we first construct the supersymmetric functional Z [9, 31]. Any correlator of the dot Green's functions can be later obtained from it by differentiation with respect to sources J [6] as long as we know the average $Z[J]$ over the γ -ensemble.

We can assume the magnetic field (equation (2)) to be vanishingly small and remove it from consideration. Its role is reduced to breaking the time reversal symmetry and justifying the use 4-component supervectors $\Psi(\mathbf{r})^T = \{S_1(\mathbf{r}), \chi_1(\mathbf{r}), S_2(\mathbf{r}), \chi_2(\mathbf{r})\}$ in the supersymmetric functional. The generalization of our formalism to other symmetry classes is quite straightforward and will require doubling the space [9]. It is convenient to express the coupling coefficients as a sum of constant and stochastic parts: $\gamma_n = \hat{\gamma}_n + \tilde{\gamma}_n$. For the statistics of $\tilde{\gamma}_n$, we assume that $\langle \tilde{\gamma}_n \rangle = 0$, $\langle \tilde{\gamma}_n \tilde{\gamma}_m \rangle = x_n^2 \delta_{nm}$; all higher moments factorize into second moments. We indicate averaging over random couplings to the leads by the shorthand notation $\langle \dots \rangle_{\tilde{\gamma}}$. Then, we 'eliminate the leads' [29], passing to the Hamiltonian given by equation (6).

In terms of supervectors $\Psi(\mathbf{r})$ and supermatrices $L = \text{diag}\{1, 1-1, 1\}$, $\Lambda = \text{diag}\{1, 1-1, -1\}$ [31], the generating functional $Z[J]$ is written down as follows:

$$\langle Z[J] \rangle_{\gamma} = \int d\Psi^* d\Psi e^{-\mathcal{L}[\Psi]} \langle e^{-\mathcal{L}_\delta[\Psi]} \rangle_{\tilde{\gamma}}, \quad (9)$$

$$\mathcal{L}[\Psi] = i \int \Psi^\dagger(\mathbf{r}) \hat{\mathcal{H}}_J L \Psi(\mathbf{r}) d\mathbf{r} + \frac{1}{2} \sum_{n=1}^N v_n \hat{\gamma}_n \int_{C_n} \Psi^\dagger(y_n) \varphi_n(y_n) \varphi_n(y'_n) \Lambda L \Psi(y'_n),$$

$$\mathcal{L}_\delta[\Psi] = \sum_{n=1}^N \frac{\tilde{\gamma}_n v_n}{2} \int_{C_n} \Psi^\dagger(y_n) \varphi_n(y_n) \varphi_n(y'_n) \Lambda L \Psi(y'_n),$$

with

$$\hat{\mathcal{H}}_J = \left(-\frac{\nabla^2}{2m} - E \right) I_4 + i\epsilon \Lambda + J,$$

where ϵ is infinitesimally small, $\varphi_n(y) = \sqrt{2/d_n} \sin(\pi y/d_n)$ (for hard-wall lead of width d_n). Here, \int_{C_n} stands for a double integration over y_n and y'_n , the transverse coordinates along the cross section C_n . The exact form of the source supermatrix,

$$J = \text{diag}\{J_1(\mathbf{r}), J_2(\mathbf{r}), J_1(\mathbf{r}), J_2(\mathbf{r})\},$$

is dictated by the choice of the physical quantity we eventually wish to calculate.

Averaging over $\tilde{\gamma}_n$ produces

$$\langle e^{-\mathcal{L}_\delta[\Psi]} \rangle_{\tilde{\gamma}} = \left\langle 1 + \frac{1}{8} \sum_{n=1}^N x_n^2 v_n^2 \left\{ \int_{C_n} \Psi^\dagger(y_n) L \varphi_n(y_n) \varphi_n^*(y'_n) \Psi(y'_n) \right\}^2 + \dots \right\rangle_{\tilde{\gamma}}. \quad (10)$$

This step towards constructing supersymmetric NL σ M for our system is no more difficult than in the problems requiring averaging over impurities or random magnetic field. As we shall see below, it results in the coupling of the effective field with the source of boundary transmission.

4. Effective problem for a smooth convex boundary

Next we carry out the Hubbard–Stratonovich transformation to decouple the ‘interaction’ terms (equation (10)) which entered the action of equation (9) after averaging. The procedure, involving supermatrix fields Q_n , is explained in appendix A and leads to

$$\begin{aligned} \langle Z[J] \rangle_\gamma = & \int \prod_n DQ_n \int d\Psi^* d\Psi e^{-\mathcal{L}[\Psi]} \exp \sum_{n=1}^N \left\{ \frac{x_n v_n m}{2} \int_{C_n} \Psi^\dagger(y_n) L \right. \\ & \times \varphi_n(y_n) \varphi_n^*(y'_n) \Psi(y'_n) \int_{C_n} Q_n(y''_n, y'''_n) \varphi_n(y''_n) \varphi_n^*(y'''_n) \\ & \left. - \frac{m^2}{2} \left(\int_{C_n} Q_n(y_n, y'_n) \varphi_n(y_n) \varphi_n^*(y'_n) \right)^2 \right\}. \end{aligned} \quad (11)$$

Next, we make further simplifications by setting $d_n = d$ and distributing the large number of narrow leads $d_n = d \ll P$ (where P is perimeter of the dot), close to each other all around the boundary. In this limit, we can approximately write $\varphi_n(y_n) \varphi_n^*(y'_n) \simeq 2/d$. These two model assumptions are not essential for subsequent analysis, although they do make it more transparent. The distribution of the leads coupling coefficients can be chosen piecewise continuous, while widths can be given as a set of input parameters to the problem at hand together with $\hat{\gamma}_n$, x_n and v_n .

Then, we make use of the mean value theorem for each of the \int_{C_n} integrals, provided all the integrands are smooth functions of y_n . The resulting sum in the exponent of equation (11) can be combined into a single integral over the arclength s , ranging from 0 to P , which runs along the boundary, defined in polar coordinates as $r = R(s)$, producing

$$\begin{aligned} \langle Z[J] \rangle_\gamma = & \int DQ(s) \int d\Psi^*(r, \theta) d\Psi(r, \theta) e^{-\mathcal{L}[\Psi]} \\ & \times \exp \left\{ 2md \int_0^P \tilde{x}(s) \Psi^\dagger(R(s), s) L Q(s) \Psi(R(s), s) ds - 2m^2 d \int_0^P Q^2(s) ds \right\}, \end{aligned}$$

where $\tilde{x}(s) = x(s)v(s)$ and both band velocity $v(s)$, mean $\hat{\gamma}(s)$ and rms $x(s)$ of the coupling strength are now smooth functions of arclength. The corresponding changes in the term $\mathcal{L}[\Psi]$ are made accordingly. Upon carrying out the Ψ -integration, we arrive at the following representation of the generating functional in polar coordinates $\mathbf{r} = (r, \theta)$:

$$\begin{aligned} \langle Z[J] \rangle_\gamma = & \int DQ \exp \text{Str} \left[-2m^2 d \int d\mathbf{r} d\mathbf{r}' Q^2(s) \delta_P \delta(\mathbf{r} - \mathbf{r}') \right. \\ & - \ln \left\{ \left(-i\hat{\mathcal{H}}_0 + \frac{v(s)\hat{\gamma}(s)}{d} \Lambda \delta_P \right) \delta(\mathbf{r} - \mathbf{r}') - \frac{md\tilde{x}(s)}{2} Q(s) \delta_P \delta(\mathbf{r} - \mathbf{r}') \right\} \\ & \left. - \ln(I_4 + \epsilon \Lambda \mathcal{G}(\mathbf{r}, \mathbf{r}') + \mathcal{G}(\mathbf{r}, \mathbf{r}') J(\mathbf{r})) \right], \end{aligned} \quad (12)$$

where δ_P is a perimeter delta-function and the supermatrix ‘Green’s function’ \mathcal{G} is determined from

$$\left\{ \hat{\mathcal{H}}_0 - i2m\tilde{x}(s)d \left(Q(s) - \frac{\hat{\gamma}(s)}{2x(s)md} \Lambda \right) \delta_P \right\} \mathcal{G}(\mathbf{r}, \mathbf{r}') = i\delta(\mathbf{r} - \mathbf{r}'), \quad (13)$$

and $\hat{\mathcal{H}}_0 = \hat{\mathcal{H}}_J(J = 0)$. Thus, we reduced the generating functional for our system to the integral over the supermatrix superfield with boundary support. We have

$$\langle Z[J] \rangle_\gamma = \int DQ \exp(F[Q] + F_J[\mathcal{G}]), \quad (14)$$

with the free energy

$$F[Q] = \text{Str} \int d\mathbf{r} d\mathbf{r}' \{-2m^2 d Q(s)^2 \delta_p \delta(\mathbf{r} - \mathbf{r}') + \ln -i\mathcal{G}^{-1}(\mathbf{r}, \mathbf{r}')\},$$

and symmetry breaking terms

$$F_J[\mathcal{G}] = -\text{Str} \int \ln(I_4 + \epsilon \Lambda \mathcal{G}(\mathbf{r}, \mathbf{r}') + \mathcal{G}(\mathbf{r}, \mathbf{r}') J(\mathbf{r})) d\mathbf{r} d\mathbf{r}'.$$

In order to reduce our generating functional (equation (14)) to a NL σ M, we employ the saddle-point condition, which in our case reads

$$Q_{\text{sp}}(s) = \frac{\tilde{x}(s)}{2m} \mathcal{G}(R(s), R(s), s, s, Q_{\text{sp}}(s)). \quad (15)$$

We assume the saddle-point solution to be diagonal: $Q_{\text{sp}}(s) = Q_0(s)\Lambda$. To proceed with the analysis of fluctuations, one needs to determine both $Q_0(s)$ and the diagonal Green's function supermatrix $\mathcal{G}_{\text{sp}}(\mathbf{r}, \mathbf{r}')$. Thus, by combining equations (13) and (15) with the assumption about saddle-point structure we mapped the original problem with random boundary condition onto an effective problem specified by the differential equation:

$$\frac{1}{2m} (\nabla^2 - \kappa^2) \mathcal{G}(r, r', \theta, \theta', \kappa^2) = -\frac{i\delta(r - r')\delta(\theta - \theta')}{r}, \quad (16)$$

where $\kappa^2 = -2mE$, with associated boundary conditions

$$\frac{\partial}{\partial r} \mathcal{G}(r, r', \theta, \theta', \kappa^2)|_{S^-} = i \frac{f(s)}{R(s)} \mathcal{G}(r, r', \theta, \theta', \kappa^2)|_{S^-}, \quad (17)$$

$$\frac{\partial}{\partial r} \mathcal{G}(r, r', \theta, \theta', \kappa^2)|_{S^+} = 0, \quad (18)$$

where $f(s) = m^2 \tilde{x}(s) \tilde{Q}_0(s) R(s) d$, S^- , S^+ are the inner and the outer surfaces of the dot and $\tilde{Q}_0(s) = Q_0(s) - \hat{\gamma}(s)/(2x(s)md)$.

5. Born–Oppenheimer-like approximation

To construct the Green's function of equation (16) we employ the technique of [32], which was also used in [5] for a circular billiard. In the latter case, the corresponding Green's function reads

$$\mathcal{G}_{\text{circle}}(r, r', \theta, \theta', \kappa^2) = \frac{im}{\pi} \sum_{n=-\infty}^{\infty} I_n(\kappa r_{<}) \{a_n I_n(\kappa r_{>}) + K_n(\kappa r_{>})\} e^{in(\theta - \theta')}, \quad (19)$$

with $r_{>}$ ($r_{<}$) is a maximum (minimum) of r and r' , I_n and K_n are modified Bessel functions, respectively, and the coefficients a_n are chosen to ensure the boundary condition. The summation in equation (19) is replaced with integration, while Bessel functions are replaced with their uniform approximations [32].

For a generic convex smooth-wall billiard, a similar expansion is possible with the help of the recently proposed Born–Oppenheimer-like approximation [26]. The method used in [26] assumes that the angular variation of $x(s)$, $f(s)$, etc is slow and enables one to determine certain classes of eigenstates quite accurately. Guided by the example of circular billiard, and the fact that WGM have the longest lifetime, we focus our attention on the limit of large angular momentum.

The Green’s function (equation (19)) was build from two radial solutions of equation (16) and angular harmonics. In the case of a generic billiard, the angular ‘part’ of the eigenstate is obtained from WKB solution of slow-variable equation [26],

$$\psi(s) = \exp \left\{ i\lambda \int^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} \right\}$$

while the radial ‘part’ is given by the linear combination of $I_{l(s)}(\kappa r_s)$ and $K_{l(s)}(\kappa r_s)$. Here, angular momentum ‘index’ is parametrized as $l(s) = \lambda F(s)$, r_s is a local radial coordinate, computed from the local centre of curvature [26]. Eigenvalues λ are determined from arclength quantization condition ($\psi(0) = \psi(P)$): $\lambda_n = 2\pi n / \left(\frac{F(s)}{R(s)} \right) P$. Here and below we use $\overline{}$ to denote the perimeter average: $\overline{R} = \int_0^P R(s) ds / P$, $\overline{f} = \int_0^P f(s) ds / P$, etc. Note that for dimensionless quantities we will not use overall factor $1/P$. The explicit form of $F(s)$ can be found from the radial quantization condition [26]. Below we consider $F(s)$ to be known. Thus, the Green’s function for the effective problem given by equations (16), (17) can be approximated as

$$\begin{aligned} \mathcal{G}(r, r', \theta, \theta', \kappa^2) &= \frac{imR(s)}{P} \sum_{n=-\infty}^{\infty} I_{\lambda_n F(s)}(\kappa r_{s<}) \{ a_n I_{\lambda_n F(s)}(\kappa r_{s>}) + K_{\lambda_n F(s)}(\kappa r_{s>}) \} \\ &\times \exp \left\{ i\lambda_n \int_{s'}^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} \right\}, \end{aligned} \tag{20}$$

with

$$a_n(s) = \frac{-if(s)K_{\lambda_n F(s)}(\kappa R(s)) + \kappa R(s)K'_{\lambda_n F(s)}(\kappa R(s))}{if(s)I_{\lambda_n F(s)}(\kappa R(s)) - \kappa R(s)I'_{\lambda_n F(s)}(\kappa R(s))}.$$

This expression does not hold far from the boundary, but it suits our purposes. For example, it can be used to determine saddle-point solution $Q_0(s)$. We have, at the billiard boundary,

$$\begin{aligned} \mathcal{G}(R(s), R(s), s, s', \kappa^2) &= \frac{imR(s)}{P} \sum_{n=-\infty}^{\infty} \frac{I_{\lambda_n F(s)}(\kappa R(s))}{if(s)I_{\lambda_n F(s)}(\kappa R(s)) - \kappa R(s)I'_{\lambda_n F(s)}(\kappa R(s))} \\ &\times \exp \left\{ i\lambda_n \int_{s'}^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} \right\}, \end{aligned}$$

Then, $Q_0(s)$ is determined by the stationary point condition (equation (15))

$$i \frac{\tilde{x}(s)R(s)}{2P} \sum_{n=-\infty}^{\infty} \frac{I_{\lambda_n F(s)}(\kappa R(s))}{if(s)I_{\lambda_n F(s)}(\kappa R(s)) - \kappa R(s)I'_{\lambda_n F(s)}(\kappa R(s))} = \tilde{Q}_0(s) + \frac{\hat{y}(s)}{2x(s)md}. \tag{21}$$

Dropping the imaginary part of $\tilde{Q}_0(s)$, we set $\tilde{g}(s) = \kappa R(s)$, and proceed as follows. We evaluate the sum over n in equation (21) asymptotically in the limit: $\tilde{g}(s) \gg 1$, $f(s)/\tilde{g}(s) \sim 1$, by replacing it with integral over corresponding continuous variable. Following the technique, described in [32], and illustrated in appendix B for a more involved calculation, we use uniform approximation for the Bessel function and its derivative [33], carry out the integration and rewrite equation (21) as

$$\frac{\tilde{x}(s)R(s)\overline{(F(s)/R(s))}}{2F(s)\sqrt{f(s)^2 - g(s)^2}} = \tilde{Q}_0(s) + \frac{\hat{y}(s)}{2x(s)md},$$

after the substitution $\tilde{g}(s) \rightarrow -ig(s)$ ($\kappa \rightarrow -ik$), to the leading order in $1/g(s)$. This equation can be now solved numerically yielding $f(s)$ (or $\tilde{Q}_0(s)$) for any prescribed set of parameters of the problem.

6. Ballistic nonlinear σ -model

Having determined the relation between the Green's function \mathcal{G} and saddle-point value of superfield Q , we come back to further analyse the generating functional specified by equation (14). As a final step in derivation of NL σ M we have to expand the action $F[Q]$ up to quadratic order around the extremum and include first variation of symmetry breaking terms in $F_J[\mathcal{G}]$. Observing that $F[Q_{\text{sp}}] = 0$, we turn to the fluctuations of Q , which decompose into a transverse piece $\delta Q^{(t)}$ (along the saddle-point manifold [9]) and a longitudinal piece $\delta Q^{(l)}$ (orthogonal to the saddle-point manifold). We focus on the transverse part of the action, the part anticommuting with the Λ -like saddle-point solution. Our goal is to demonstrate that in the absence of symmetry breaking terms transverse fluctuations are massless (Goldstone) modes of the theory. The purely transverse terms are given by [9]

$$F_t[\delta Q] = -4m^2 d \text{Str} \int_0^P (\delta Q^{(t)}(s))^2 ds + (md)^2 \text{Str} \int_0^P \int_0^P \mathcal{G}(Q_0(s)) \mathcal{G}(-Q_0(s)) \\ \times \tilde{x}(s) \tilde{x}(s') \delta Q^{(t)}(s) \delta Q^{(t)}(s') ds ds'.$$

In view of the developments of the previous section, we can expand deviations of Q in approximate angular eigenstates: $\delta Q^{(t)}(s) = \sum_{l=-\infty}^{\infty} Q_l^{(t)} \exp\{i\lambda_l \int^s d\bar{s} F(\bar{s}/R(\bar{s}))\}$, and, setting $\mathcal{G}(Q_0) = \mathcal{G}$, and $\mathcal{G}(-Q_0) = \tilde{\mathcal{G}}$, we obtain

$$F_t[\delta Q] = -4m^2 d \text{Str} \sum_{l,n} \int_0^P \delta Q_l^{(t)} \delta Q_n^{(t)} \exp\left(i(\lambda_l + \lambda_n) \int^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s}\right) ds \\ + (md)^2 \text{Str} \sum_{l,n} \delta Q_l^{(t)} \delta Q_n^{(t)} \int_0^P ds \int_0^P ds' \mathcal{G} \tilde{\mathcal{G}} \tilde{x}(s) \tilde{x}(s') \\ \times \exp\left(i\lambda_l \int^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} + i\lambda_n \int^{s'} \frac{F(\bar{s})}{R(\bar{s})} d\bar{s}\right). \quad (22)$$

Our intermediate goal now is to get a Ward-like identity, which would allow us to evaluate massive part of the transverse action. At this point, we consider equation (13) for both Green's functions

$$\left(\frac{\nabla^2}{2m} - i\frac{f(s)}{2mR(s)}\delta_P - \frac{\kappa^2}{2m}\right)\mathcal{G} = \frac{i}{r}\delta(r-r')\delta(\theta-\theta'), \quad (23)$$

$$\left(\frac{\nabla^2}{2m} + i\frac{f(s)}{2mR(s)}\delta_P - \frac{\kappa^2}{2m}\right)\tilde{\mathcal{G}} = \frac{i}{r}\delta(r-r')\delta(\theta-\theta'), \quad (24)$$

and multiply equation (23) by $\tilde{\mathcal{G}}h(s, s')$ and equation (24) with $\mathcal{G}h(s, s')$. We are free to choose function $h(s, s')$ to behave as we like inside the billiard, because only the value of this function at the boundary matters for manipulations in equation (22). In particular, we may require that h slowly decay to zero in the radial variable as we move away from the boundary. Since the radial rate of variation is approximately $1/\bar{R}$, while the angular rate is $1/\lambda_F$ (where λ_F is a Fermi wavelength), we neglect all radial derivatives in the subsequent analysis, which is correct to leading order in \bar{R}/λ_F . Next, we subtract one equation from the other, integrate the resulting expression over the area of the dot D (including the boundary: $0 < r < R^+(s)$) and set the remaining free radial coordinate to $R(s)$ to get

$$\int_V (h\tilde{\mathcal{G}}\nabla \cdot \nabla \mathcal{G} - h\mathcal{G}\nabla \cdot \nabla \tilde{\mathcal{G}}) d\tau = 2i \int_0^P ds f(s) h(s, s') \\ \times \mathcal{G}(R(s), R(s'), s, s') \tilde{\mathcal{G}}(R(s), R(s'), s, s') + 2imh(s', s') \\ \times (\tilde{\mathcal{G}}(R(s'), R(s'), s', s') - \mathcal{G}(R(s'), R(s'), s', s')). \quad (25)$$

Note that according to the saddle-point condition (equation (15))

$$\tilde{\mathcal{G}}(R(s), R(s), s, s) - \mathcal{G}(R(s), R(s), s, s) = -2m \frac{\tilde{Q}_0(s)}{\tilde{x}(s)}, \quad (26)$$

which might be used in the last line in equation (25). Furthermore, one can transform the left-hand side of the equation (25) by making use of Gauss's theorem for the domain D (with boundary ∂D)

$$\begin{aligned} & \int_D (h\tilde{\mathcal{G}}\nabla \cdot \nabla \mathcal{G} - h\mathcal{G}\nabla \cdot \nabla \tilde{\mathcal{G}}) d\tau \\ &= \int_{\partial D} h(\tilde{\mathcal{G}}\nabla \mathcal{G} - \mathcal{G}\nabla \tilde{\mathcal{G}}) \cdot d\sigma - \int_D (\nabla(h\tilde{\mathcal{G}}) \cdot \nabla \mathcal{G} - \nabla(h\mathcal{G}) \cdot \nabla \tilde{\mathcal{G}}) d\tau. \end{aligned} \quad (27)$$

Application of the boundary conditions (equation (18)) to equation (27) makes the surface integral vanish. It remains to evaluate the area integral on the right-hand side of equation (27), which we now substitute into equation (25) and obtain, neglecting the terms with radial derivatives of h ,

$$\int_0^{2\pi} \frac{\partial h(s, s')}{\partial \theta} d\theta \int_0^{R(s)} \frac{dr}{r} \left(\tilde{\mathcal{G}} \frac{\partial}{\partial \theta} \mathcal{G} - \mathcal{G} \frac{\partial}{\partial \theta} \tilde{\mathcal{G}} \right) + 4im^2 h(s', s') \frac{\tilde{Q}_0(s)}{\tilde{x}(s)} = 2i \int_0^P ds h(s, s') \tilde{\mathcal{G}} \mathcal{G}, \quad (28)$$

where we used equation (26). The boundary value of function h is completely at our discretion. We select

$$h(s, s') = -i(md)^2 \sum_{l,n} \frac{\tilde{x}(s)\tilde{x}(s')}{f(s)} \exp \left(i\lambda_l \int^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} + i\lambda_n \int^{s'} \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} \right) \delta Q_l^{(t)} \delta Q_n^{(t)}.$$

Then we integrate both sides of equation (28) with respect to s' along the boundary of the billiard and substitute the result together with h in each term of the sum in the second term of $F_l[\delta Q]$ (equation (22)).

We observe that the massive contribution coming from variation of $Q(s)^2$ (the first term on the right-hand side of equation (22)) will be cancelled by the second term on the left-hand side of equation (28). The surviving linear in $\delta Q^{(t)}$ term proportional to $\hat{\gamma}(s)$ (cf [5]) will be accounted later, together with the symmetry breaking terms. In addition to that, we need to demonstrate that massive longitudinal modes $\delta Q^{(l)}$ are decoupled from the essentially massless transverse modes. Then, the longitudinal fluctuations around saddle point can be all integrated out, producing unity due to supersymmetry.

To complete the derivation of NL σ M we still have to analyse the remaining part of the transverse action

$$F_l[\delta Q] = \text{Str} \sum_{l,n} \int_0^P ds' \int_0^P ds \int_0^{R(s)} \frac{\partial h(s, s')}{\partial s} R^2(s) \left(\tilde{\mathcal{G}} \frac{\partial}{\partial s} \mathcal{G} - \mathcal{G} \frac{\partial}{\partial s} \tilde{\mathcal{G}} \right) \frac{dr}{r}. \quad (29)$$

We perform this step using the asymptotic technique, we employed in previous section. After a straightforward but lengthy procedure, which we allocate to appendix B, we obtain

$$\begin{aligned} F_l[\delta Q] &\simeq -\text{Str} D_0 \int_0^P \left(\frac{R(s)}{F(s)} \frac{\partial \delta Q^{(t)}}{\partial s} \right)^2 ds, \\ D_0 &= \left(\frac{F}{R} \right) \frac{m^4 d^2 \bar{x}}{4P} \left(\frac{\tilde{x} F R^2 g (2g^2 + f^2)}{f^4 \sqrt{f^2 - g^2}} \right). \end{aligned} \quad (30)$$

One can use similar set of manipulations for the purely longitudinal and proportional to $\delta Q^{(l)} \delta Q^{(t)} + \delta Q^{(t)} \delta Q^{(l)}$ parts of the action, which consist of the fluctuations commuting with

the saddle-point solution, and therefore contain \mathcal{G}^2 in the contribution to second variation coming from the logarithmic term in $F[Q]$ (cf equation (22)). These manipulations enable us to show that to the considered order in $1/\bar{g}$ the longitudinal and transverse modes are decoupled (the coefficient in front of the $\delta Q^{(l)}\delta Q^{(t)} + \delta Q^{(t)}\delta Q^{(l)}$ is small compared to the mass of the longitudinal modes).

To finish the construction of the nonlinear σ -model, we integrate out the longitudinal modes, set $\delta Q^{(l)} = Q$, and expand the symmetry breaking terms $F_J[\mathcal{G}(Q)]$ to the lowest order in J and ϵ to get

$$\langle Z[J] \rangle_\gamma = \int DQ e^{-F[Q]},$$

with the free energy given by

$$F[Q] = \text{Str} \int d\mathbf{r} d\mathbf{r}' \left[\left\{ D_0 \left(\frac{R}{F} \frac{\partial Q(s)}{\partial s} \right)^2 + \frac{\bar{\gamma}}{xmd} Q(s) \Lambda \right\} \delta_P \delta(\mathbf{r} - \mathbf{r}') \right. \\ \left. + \int ds'' (\epsilon \Lambda + J(\mathbf{r})) Q(s'') a(\mathbf{r}, R(s'')s''; \mathbf{r}', R(s'')s'') \right] \quad (31)$$

where

$$a(\mathbf{r}, R(s'')s''; \mathbf{r}', R(s'')s'') = i \frac{md\bar{x}}{2} \mathcal{G}_{\text{sp}}(R(s'')s'', \mathbf{r}') \mathcal{G}_{\text{sp}}(\mathbf{r}, R(s'')s'')$$

and we suppressed explicit slow s -dependence everywhere. The free energy given by equation (31) is the central result of our paper. It displays the surface modes $Q(s)$, which undergo diffusion and drift, and are coupled to the interior of the dot by the last term.

For the calculation of any physical quantity, expressed via Green's function correlators such as, e.g. $\langle G^R(\mathbf{r}, \mathbf{r}') G^A(\mathbf{r}, \mathbf{r}') \rangle_\gamma$, it is necessary to have actual parametrization for $Q(s)$. Since basic symmetries of our problem are not any different from these of the corresponding $Q(\mathbf{r})$ -field of the diffusive problem [6, 9, 31] our supermatrix Q can be parametrized as suggested in [9, 31].

7. Discussion

We have constructed a non-perturbative framework to analyse one particular realization of a whole class of nanostructures: a nearly closed system with ballistic internal dynamics and random losses at the boundary. Our approach uses a natural regularizer, which enables us to circumvent the technical difficulties of previous approaches to closed ballistic systems [10, 23–25]. We find that the resulting theory can be characterized by diffusive modes confined to the boundary and interacting nonlocally with the interior as encapsulated in our main result (equation (31)). These diffusive modes are identified as WGM, which are exponentially long-lived compared to other trajectories scattering off the boundary non-tangentially and, therefore, are anticipated to dominate the long-time behaviour of response functions. We expect our formalism to be useful whenever a ballistic nanostructure supports such modes. Note that no RMT assumptions have been made, and our approach is applicable in a broad energy range and includes the spatial dependence of physical quantities.

We derive our results by treating a ballistic billiard as a scattering system. In particular, we use an effective non-Hermitian Hamiltonian, which is a common tool in quantum chaotic scattering. However, the resulting supersymmetric NL σ M is fully intended for the calculations of 'internal' characteristics, such as local density of states statistic or correlators

of eigenfunctions, rather than statistics of reflection or transmission. For that reason, the escape rate is chosen to be small and distributed along the boundary. In addition, we chose the coupling part of the non-Hermitian Hamiltonian to be fluctuating, thus defining an ensemble of billiards with different escape rates. These steps allow us to study ballistic nanostructures without the introduction of any other regularizer [6, 24]. The simplest realization of such model is a circular billiard with constant parameters, for which the surface diffusion NL σ M was obtained in [5] using the saddle-point solution. This billiard is also the classical example of a system supporting WGM.

In this paper, we have extended the results of [5] to show that surface diffusion modes are present in generic, chaotic, smooth-wall billiard as well. To create an extension to other chaotic optical and electronic systems, for which WGM are of importance [12, 19–21], one needs to adapt our procedure to the details of the coupling to the environment. However, for most mesoscopic structures the wall absorption, which is always present in experiment, can be adequately modelled by uniformly distributed leads. Furthermore, our model allows us to add finite number of wide leads if the problem under consideration involves transmission properties. Other calculations within a similar framework can be performed for systems which have disorder concentrated at the boundary, e.g. quantum corral composed of different atoms or artificial atoms proposed in [21] with the random magnitude of the magnetic flux.

As for handling the calculation of correlators of physical quantities one needs to introduce a difference between two energies ω into the derivation. For the final expression for the free energy (equation (31)) this means a replacement $\epsilon \rightarrow \epsilon + i\omega/2$. A good starting point can be the application the perturbative technique described in [37] to the analysis of the 2-point function in a circular cavity. Assuming $Q(\theta)$ (θ being the polar angle) to fluctuate weakly near the origin $Q_0\Lambda$ [5], one can decompose the supermatrix $Q(\theta)$ as $T^{-1}\tilde{Q}(\theta)T$, where T are angle-independent pseudounitary supermatrices, which are, in turn, parametrized in terms of off-diagonal supermatrices W [9, 37]. The perturbation scheme is based on the expansion of $Q(\theta)$ around the saddle-point solution ($Q_0\Lambda$) in terms of W (see the details in [37]).

It is also natural to enquire about connection of our model to the existing ballistic analogues of the diffusive NL σ Ms, e.g. the models proposed in [6, 10, 23]. Note that the zero-dimensional versions of these theories produce RMT, which makes the answer to this question important for studies of truly ergodic systems. The construction involving diffuse boundary scattering ([10]) is essentially different from ours, because in [10] the electron loses memory after a single boundary collision, while in our model, the WGM trajectories retain phase coherence until the electron finally leaves the system. Our treatment is complementary (only in conceptual sense) to that of [6, 23], in which only modes of Q inside the ballistic dot appear, and the boundary value of Q is nonfluctuating. The zero-dimensional case $Q(\mathbf{r}) = \text{const.}$ inside the dot ($Q(\theta) = \text{const.}$ in our case), i.e. the situation in which only lowest mode contributes, seems to be the only possible connection between these models and ours. However, another crucial difference between the two formulations is worth mentioning. The universal parameter τ (mean free time between collisions) in the Muzykantskii–Khmelnitskii σ -model ([23]) is angular momentum dependent in our case. In order to address this issue of zero-dimensional correspondence between our result and those of [6, 23] one needs to carry out the calculations for correlation function of a chaotic system at hand using the NL σ M we derived and compare it to the RMT result in the appropriate limit. This is a subject of ongoing work to be presented elsewhere.

Finally, we hope to consider applications of our NL σ M to the interacting-electron problem in the future. One of the possible ways to take the interactions into account in diffusive and ballistic systems with large dimensionless conductance is to use a ‘Universal Hamiltonian’ [34], which was shown to be the renormalization group fixed point for weak interactions

[22, 35]. We hope to extend our analysis to the interacting ballistic case by using the large- N approach of [22].

Acknowledgment

We are grateful to the NSF for partial support under DMR-0311761.

Appendix A. Hubbard–Stratonovich decoupling

In this appendix, we derive the identity which decouples supervector variables in the action averaged over $\tilde{\gamma}_n$.

For arbitrary function $V(u, z)I_4$ which does not possess any supersymmetric structure and for two coordinate-dependent supermatrices $Q(u, z)$ and $A(u, z)$ which have the same supersymmetric structure and, therefore commute, the following generalization of the decoupling rule can be verified:

$$\int DQ \exp\left(-\text{Str} \iint \alpha QVQV - i\beta QVAV\right) = \exp\left(-\frac{\beta^2}{4\alpha} \text{Str} \iint AVAV\right), \quad (\text{A.1})$$

where we skipped coordinate dependence on y_n and y'_n and corresponding differentials. First of all, we make use of the theorem due to Parisi–Sourlas–Efetov–Wegner (see, for example, [36]), which states that for any ‘invariant superfunction’ the corresponding superintegral is equal to its value at the origin, which in our case is unity. By shifting $Q \mapsto Q + i\delta A$ and setting $2\alpha\delta - \beta = 0$, we get

$$\begin{aligned} -\alpha \int_{C_n} QV \int_{C_n} QV + i\beta \int_{C_n} QV \int_{C_n} AV &\mapsto -\alpha \int_{C_n} QV \int_{C_n} QV - i(2\alpha\delta - \beta) \\ &\times \int_{C_n} AV \int_{C_n} QV = -\alpha \iint_{C_n} QVQV - \frac{\beta^2}{4\alpha} \iint_{C_n} AVAV. \end{aligned} \quad (\text{A.2})$$

Then, we carry out the Hubbard–Stratonovich transformation by applying equation (A.1) from right to left to each of the terms in equation (10) (in our case $-\beta^2/4\alpha = x_n^2 v_n^2/8$ and $V(y_n, y'_n) = \varphi_n(y_n)\varphi_n^*(y'_n)$). We choose $\alpha = m^2/2$, so that $i\beta = mx_n v_n/2$ and get equation (11).

Appendix B. Diffusion term

Approximating the Green functions $\tilde{\mathcal{G}}(r, R(s'), s, s')$ and $\mathcal{G}(r, R(s'), s, s')$ according to equation (20) and h with its boundary value, we carry out r -integration with the help of the identity

$$\int \frac{I_\alpha(z)I_\beta(z)}{z} dz = z \frac{I'_\alpha(z)I_\beta(z) - I_\alpha(z)I'_\beta(z)}{\alpha^2 - \beta^2}$$

and get

$$\begin{aligned} F_l[\delta Q] &= i \left(\frac{m^2 d}{P}\right)^2 \sum_{l,n,p,q} \int_0^P ds' \int_0^P ds \frac{\tilde{x}(s)\tilde{x}(s')g(s)D_{p,q}(\tilde{g}(s))\lambda_n}{f(s)(\lambda_p + \lambda_q)} \\ &\times \exp\left(i\lambda_l \int^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} + i\lambda_n \int^{s'} \frac{F(\bar{s})}{R(\bar{s})} d\bar{s} + i(\lambda_p + \lambda_q) \int_{s'}^s \frac{F(\bar{s})}{R(\bar{s})} d\bar{s}\right) \delta Q_l^{(t)} \delta Q_n^{(t)}, \end{aligned}$$

where

$$D_{p,q} = 1/(\mathfrak{j}_{p,q}/\mathfrak{w}_{p,q} + i f(s)\tilde{g}(s))$$

with

$$\mathfrak{j}_{p,q} = f^2 I_{\lambda_p F}(\tilde{g}) I_{\lambda_q F}(\tilde{g}) + \tilde{g}^2 I'_{\lambda_p F}(\tilde{g}) I'_{\lambda_q F}(\tilde{g}),$$

$$\mathfrak{w}_{p,q} = I_{\lambda_p F}(\tilde{g}) I'_{\lambda_q F}(\tilde{g}) - I'_{\lambda_p F}(\tilde{g}) I_{\lambda_q F}(\tilde{g}).$$

where we omitted s -dependence for brevity in the last two lines; we continue to do that throughout this appendix. Integration over s and s' in the above expression for $F_t[\delta Q]$ produces infinitesimally small result unless $n + p + q = 0, l = -n$. Anticipating these averages we concentrate on the evaluation of the sum over p and q restricted by these two conditions. Later on one can approximate $F_t[\delta Q]$ by replacing the result of the summation (together with the rest of the integrand) with its perimeter averaged value. With this in mind we manipulate $\sum_{l,n,p,q}$ into a double sum

$$\begin{aligned} \sum_{l,n,p,q} D_{pq} \delta Q_l^{(t)} \delta Q_n^{(t)} &\mapsto \sum_{l,n=-\infty}^{\infty} D_{|n|,|l+n|} \delta Q_l^{(t)} \delta Q_{-l}^{(t)} \\ &\mapsto 2 \sum_{|l|} \delta Q_{|l|}^{(t)} \delta Q_{-|l|}^{(t)} \sum_{|n|} (D_{|n|,|n+|l|} + D_{|n|,|n-|l|}). \end{aligned}$$

The sum over $|n|$ can be performed asymptotically in the limit: $\tilde{g}(s) \gg 1, f(s)/\tilde{g}(s) \sim 1$ [32]. We convert this sum into the integral over the new variable

$$\mu = \frac{2\pi |n| F(s)}{\tilde{g}(s) P(F/R)},$$

use uniform expansion for the Bessel function $I_{\lambda_{|n|} F(s)}(\lambda_{|n|} F(s)/\mu)$ [33] and expand $\mathfrak{j}_{|n|,|n+|l|}/\mathfrak{w}_{|n+|l|}$ and $\mathfrak{j}_{|n|,|n-|l|}/\mathfrak{w}_{|n-|l|}$ to the leading order in $1/\tilde{g}$. We have

$$\begin{aligned} \sum_{|n|} (D_{|n|,|n+|l|} + D_{|n|,|n-|l|}) &= \frac{PF}{2\pi} \int_0^\infty d\mu \frac{(1 - \mu^2 - 2\mu^4 + \frac{f^2}{\tilde{g}^2}) \lambda_n^2}{\tilde{g}(1 + \mu^2)^{3/2} (1 + \mu^2 + \frac{f^2}{\tilde{g}^2})} \\ &\quad - i \frac{PF}{2\pi} \int_0^\infty d\mu \frac{(1 + f\tilde{g}) \mu^2 \lambda_n^2}{\tilde{g}(1 + \mu^2) (f^2 + \tilde{g}^2 (1 + \mu^2))^2}. \end{aligned} \quad (\text{B.1})$$

Then, the first integral with respect to μ in equation (B.1) vanishes, while the second one yields $(\pi/4 f^4 g) \{ (f^2 + 2g^2)/\sqrt{f^2 - g^2} - 2g \}$ after we substitute $\tilde{g}(s) \rightarrow -ig(s)$. To arrive at equation (30) we drop some of the terms which are negligible in the above-mentioned limit.

References

- [1] Davies J H and Long A R (ed) 1992 *Physics of Nanostructures* (Edinburgh: SUSSP)
- [2] Stöckmann H J 1999 *Quantum Chaos: An Introduction* (Cambridge: Cambridge University Press)
- [3] Berry M V 1981 *Eur. J. Phys.* **2** 91
- [4] Jalabert R A, Baranger H U and Stone A D 1990 *Phys. Rev. Lett.* **65** 2442
Lin W A and Jensen R V 1996 *Phys. Rev. B* **53** 3638
- [5] Rozhkov I and Murthy G 2005 *Preprint cond-mat/0504653*
- [6] Altland A, Offer C R and Simons B D 1999 *Supersymmetry and Trace Formulae* ed I V Lerner *et al* (Dordrecht: Kluwer)
- [7] Stone A D 1992 *Physics of Nanostructures* ed J H Davies and A R Long (Edinburgh: SUSSP)
- [8] Lee P A and Ramakrishnan T V 1985 *Rev. Mod. Phys.* **57** 287
Kramer B and MacKinnon A 1993 *Rep. Prog. Phys.* **56** 1469
- [9] Efetov K 1997 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press)

- [10] Blanter Y M, Mirlin A D and Muzykantskii B A 1998 *Phys. Rev. Lett.* **80** 4161
Tripathi V and Khmelnitskii D E 1998 *Phys. Rev. B* **58** 1122
Blanter Y M, Mirlin A D and Muzykantskii B A 2001 *Phys. Rev. B* **63** 235315
- [11] Marcus C M *et al* 1992 *Phys. Rev. Lett.* **69** 506
Kouwenhoven L P *et al* 1997 *Proc. NATO ASI on Mesoscopic Electronic Transport* ed L L Sohn, G Sohn and L P Kouwenhoven (Dordrecht: Kluwer) 105
- [12] Fiete G A and Heller E J 2003 *Rev. Mod. Phys.* **75** 933
- [13] Schwieters C D, Alford J A and Delos J B 1996 *Phys. Rev. B* **54** 10652
- [14] Fyodorov Y V and Sommers H J 1997 *J. Math. Phys.* **38** 1918
Dittes F M 2000 *Phys. Rep.* **339** 215
- [15] Verbaarschot J M, Weidenmüller H A and Zirnbauer M R 1985 *Phys. Rep.* **129** 367
- [16] Fyodorov Y V and Sommers H J 1996 *JETP Lett.* **63** 970
Fyodorov Y V, Khoruzhenko B A and Sommers H J 1997 *Phys. Lett. A* **226** 46
- [17] Lobkis O I, Rozhkov I S and Weaver R L 2003 *Phys. Rev. Lett.* **91** 194101
- [18] Lewenkopf C H, Müller A and Doron D E 1992 *Phys. Rev. A* **45** 2635
Pichugin K, Schanz H and Seba P 2001 *Phys. Rev. E* **64** 056227
Mendez-Sanchez R A *et al* 2003 *Phys. Rev. Lett.* **91** 174102
- [19] des Francs G C *et al* 2001 *Phys. Rev. Lett.* **86** 4950
- [20] Nöckel J U and Stone A D 1997 *Nature* **385** 45
Turechi H E *et al* 2003 *Preprint physics/0308016*
- [21] Decanini Y and Folacci A 2003 *Phys. Rev. A* **67** 042704
- [22] Murthy G *et al* 2004 *Phys. Rev. B* **69** 075321
- [23] Muzykantskii B A and Khmelnitskii D E 1995 *Pis'ma Zh. Eksp. Teor. Fiz.* **62** 68
Muzykantskii B A and Khmelnitskii D E 1995 *JETP Lett.* **62** 76
Andreev A V, Simons B D, Agam O and Altshuler B L 1996 *Nucl. Phys. B* **482** 536
- [24] Aleiner I L and Larkin A I 1996 *Phys. Rev. B* **54** 14423
- [25] Efetov K B and Kogan V R 2003 *Phys. Rev. B* **67** 245312
Efetov K B, Schwiete G and Takahashi K 2004 *Phys. Rev. Lett.* **92** 026807
- [26] Zaitsev O, Narevich R and Prange R E 2001 *Found. Phys.* **31** 7
- [27] Mello P A and Baranger H U 1999 *Waves Random Media* **9** 105
Alhassid Y 2000 *Rev. Mod. Phys.* **72** 895
- [28] Zirnbauer M R 1992 *Phys. Rev. Lett.* **69** 1584
- [29] Efetov K 1997 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press) chapter 9
- [30] Rozhkov I, Fyodorov Y V and Weaver R L 2003 *Phys. Rev. E* **68** 016204
Rozhkov I, Fyodorov Y V and Weaver R L 2004 *Phys. Rev. E* **69** 036206
- [31] Mirlin A D 2000 *Phys. Rep.* **326** 259
- [32] Stewartson K and Waechter R T 1971 *Proc. Camb. Phil. Soc.* **69** 353
Sieber M *et al* 1995 *J. Phys. A: Math. Gen.* **28** 5041
- [33] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
- [34] Brouwer P W, Oreg Y and Halperin B I 1999 *Phys. Rev. B* **R13977**
Baranger H U, Ullmo D and Glazman L I 2000 *Phys. Rev. B* **61** R2425
Kurland I L, Aleiner I L and Altshuler B L 2000 *Phys. Rev. B* **62** 14886
- [35] Murthy G and Mathur H 2002 *Phys. Rev. Lett.* **89** 126804
- [36] Haake F 2001 *Quantum Signatures of Chaos* 2nd edn (Berlin: Springer)
- [37] Kravtsov V E and Mirlin A D 1994 *JETP Lett.* **60** 656